Stochastic Process Final Exam

1 Customers at a Store

The arrival of customers that come to a store in New Haven can be modeled as an inhomogeneous Poisson process. This store opens at 8am. The arrival rate of customers is 5 people per hour at 8am. The arrival rate increases linearly from 8am to 11am and reaches the peak rate, 20 people per hour. From 11am to 1pm, the arrival rate stays at 20 people per hour. From 1pm to 5pm, the arrival rate decreases linearly and becomes 12 people per hour at 5pm.

1. Find the probability that no customer comes between 8:30am and 9:30am.

2. Find the expected number of customers that come to this store between 8:30am and 1:30pm.

2 Yale Bulldogs T-Shirts

Every time Yale Bulldogs T-shirts are sold out at the Yale Bookstore, the staff will purchase one thousand new T-shirts for sale. The amount of time (in days) that a purchase process takes is uniformly distributed over $[8, 14]$ and the time (also in days) for the one thousand T-shirts to be sold out is exponentially distributed with mean 60 days. How often do the staff have to purchase another thousand T-shirts on average?

3 Utility Cost

A public utility will be installed at Yale. The installment cost is c_0. The lifetime of this utility is random with cumulative distribution function F and it will be replaced either upon a failure or at age T. When it is replaced, Yale only needs to pay the replacement cost rather than the installment cost c_0. The replacement cost will be c_1 at age T and c_2 upon a failure. What is the long-run cost per unit time?

4 Convergence of Random Variables

Let X_1, X_2, ... be independent continuous random variables. The probability density function of X_n is

$$f_n(x) = \frac{n}{\pi(1 + n^2 x^2)}.$$

Does the sequence of random variables $\{X_n \colon n \in \mathbb{N}\}$ converge almost surely, in L^p $(p \geqslant 1)$, in probability or in distribution?

5 Markov Chain

5.1 Classification of States

Suppose that a Markov chain has the state space $S = \{1, 2, ..., 6\}$ and the transition matrix

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.4 & 0.6 & 0 & 0 & 0 & 0 \\ 0.3 & 0 & 0.4 & 0.2 & 0.1 & 0 \\ 0 & 0 & 0 & 0.3 & 0.7 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 0 & 0.3 & 0 & 0.7 \end{pmatrix}.$$

1. We learned in class that the state space S can be partitioned/decomposed uniquely as $S = T \cup C_1 \cup C_2 \cup \cdots$, where T is the set of transient states and C_i's are closed and irreducible sets of states. Show the decomposition of the state space of this Markov chain.

2. Determine whether each state is transient, null recurrent, or positive recurrent and find its period.

3. Let τ_j be the mean first passage time of state $j \in S$. Find τ_2 and τ_5. (Hint: each closed and irreducible set of states can be viewed as a Markov chain itself; you may want to compute τ_j via the steady state distribution.)

5.2 Random Rook and Bishop

Consider a rook and a bishop both sitting on the lower left corner square of a 4×4 chess board (rather than a 8×8 chess board!). They met the same witch that the knight met in Problem Set 4, so they both start to perform independent random walks with synchronous steps on the chess board; at each step, they are equally likely to make any one of the possible moves independent of each other and in a synchronous manner. Find the expected number of steps until they meet again at the corner square where they start. (Hint: a bishop can only walk across half of the squares! Consider a product state space for the rook and bishop.)

6 Conditional Expectation

Let $\Omega = \{1, 2, 3, ..., 6\}$ and $P(\{i\}) = 1/6, \forall 1 \leqslant i \leqslant 6$. Let $\mathcal{F} = 2^{\Omega}$. Thus $(\Omega, \mathcal{F}, P)$ is a probability space. Let $\mathcal{F}_1 = \{\varnothing, \Omega\}$, $\mathcal{F}_2 = \sigma(\{1, 2, 3\})$, and $X\colon \Omega \to \mathbb{R}$ satisfying $X(i) = \max\{i - 4, 0\}$. Find $\sigma(X)$, $E(X|\mathcal{F}_1)$, $E(X|\mathcal{F}_2)$ and $E(X|\mathcal{F})$.

7 Martingale

Let $X_1, X_2, \ldots$ be i.i.d. random variables satisfying $P(X_n = 1) = P(X_n = -1) = 1/2$ for all n. Let $S_0 = 0$; for $n > 0$, define $S_n = \sum_{i=1}^{n} X_i$. For $n \geqslant 0$, define $Y_n = S_n^2 - n$.

1. Show that $\{Y_n : n \in \mathbb{N} \cup \{0\}\}$ is a martingale with respect to $\{S_n : n \in \mathbb{N} \cup \{0\}\}$.

2. Fix two integers $a > 0$ and $b < 0$. Define

$$J = \inf\{n \in \mathbb{N} \cup \{0\} : S_n = a \text{ or } S_n = b).$$

   Show that J is a stopping time.

3. Apply the optional stopping theorem to $\{Y_n : n \in \mathbb{N} \cup \{0\}\}$ and find $E[J]$.