Stochastic Process Problem Set 1

1 σ-field

(1) Let the fundamental set $\Omega = \{1, 2, 3\}$. Explicitly enumerate all possible σ-fields of Ω.

(2) Let $\Omega = \{1, 2, 3, 4\}$ and $\mathcal{E} = \{\{1\}, \{2\}\}$. Explicitly describe the σ-field $\sigma(\mathcal{E})$ generated by $\mathcal{E}$.

(3) Let $\Omega = \mathbb{R}$ and $\mathcal{A} = \{A \subseteq \Omega: A \text{ is a finite set or } A^c \text{ is a finite set}\}$. Is $\mathcal{A}$ a σ-field? Explain why.

(4) Let $\Omega = \mathbb{R}$ and $\mathcal{A} = \{A \subseteq \Omega: A \text{ is a countable set or } A^c \text{ is a countable set}\}$. Is $\mathcal{A}$ a σ-field? Explain why.

(5) (Very easy question. Don't be afraid.) Let $(\Omega, \mathcal{F})$ be a measurable space, where $\mathcal{F}$ is the σ-field. Let $A_1, A_2, A_3, \ldots$ be a sequence of events and $A_i \in \mathcal{F}$ for all i. Define

$$\operatorname*{limsup}_{n \to \infty} A_n = \bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} A_n,$$

$$\operatorname*{liminf}_{n \to \infty} A_n = \bigcup_{k \geqslant 1} \bigcap_{n \geqslant k} A_n.$$

The set $\mathrm{limsup}_{n\to\infty} A_n$ comprises all elements that appear in infinitely many numbers of A_n's. To see this, you may want to translate the set-theoretic language into the logical language; i.e.,

$$x \in \bigcap_{k \geqslant 1} \bigcup_{n \geqslant k} A_n$$

if and only if for all $k \geqslant 1$ ($\bigcap$ corresponds to $\forall$), there exists some $n \geqslant k$ ($\bigcup$ corresponds to $\exists$) such that $x \in A_n$. This actually means that x belongs to infinitely many number of A_n's. In light of this example, can you describe what $\mathrm{liminf}_{n\to\infty} A_n$ is? Are $\mathrm{liminf}_{n\to\infty} A_n$ and $\mathrm{limsup}_{n\to\infty} A_n$ $\mathcal{F}$-measurable?

2 Probability Measure

Let $\Omega = [0, 1]$ and P be the Lebesgue measure on $[0, 1]$.

(1) Compute $P(\mathbb{Q} \cap [0, 1])$, where $\mathbb{Q}$ is the set of all rational numbers and $\mathbb{Q} \cap [0, 1]$ is the set of all rationals numbers in $[0, 1]$.

(2) Compute the Lebesgue measure of the set of all irrationals in $[0, 1]$ (denoted by I).

(3) Lin, a TA of this course, used the following way to compute the Lebesgue measure of irrationals in $[0, 1]$: $P(I) = \sum_{x \in I} P(\{x\})$. Since the Lebesgue measure of $\{x\}$ is 0, then $P(I) = \sum_{x \in I} 0 = 0$. Is his calculation right? Explain why.

(4) (Not very hard with the hint) Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $A_1, A_2, A_3, \ldots$ be a sequence of events and $A_i \in \mathcal{F}$ for all i.

1. Show that

$$P\left(\bigcap_{n \geqslant k} A_n\right) \leqslant \inf_{n \geqslant k} P(A_n).$$

Hint: the probability of intersection of events is no greater than the probability of any single of them; i.e.,

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) \leqslant P(A_j), \forall j = 1, 2, 3, \ldots$$

2. Show that

$$P\left(\liminf_{n\to\infty} A_n\right) \leqslant \lim_{k\to\infty} \inf_{n\geqslant k} P(A_n).$$

We also write $\lim_{k\to\infty} \inf_{n\geqslant k} P(A_n)$ as $\liminf_{n\to\infty} A_n$. Thus you have shown that

$$P\left(\liminf_{n\to\infty} A_n\right) \leqslant \liminf_{n\to\infty} P(A_n).$$

Hint: $B_k = \bigcap_{n\geqslant k} A_n$ is an increasing set sequence. Thus

$$P\left(\bigcup_{k\geqslant 1} B_k\right) = \lim_{k\to\infty} P(B_k).$$

And note that

$$\bigcup_{k\geqslant 1} B_k = \liminf_{n\to\infty} A_n.$$

3 Poisson Process

(1) Let's model the number of cars that pass the crossroad of Grove Street and Temple Street as a Poisson process. According to the traffic statistics, the probability that no car passes this crossroad in one minute is 0.2. What is the probability that more than one car passes this crossroad in two minutes?

(2) Let $\{N(t): t \geqslant 0\}$ be a Poisson process with rate $\lambda = 4$. Find

$$\mathbb{E}[(N(4) - N(2))(N(3) - N(1))].$$

Justify your answer.

4 Convergence of Random Variables

(1) Suppose that the random variable U is uniform on $[0, 1]$ and define

$$X_n = \begin{cases} n, & \text{if } U \leqslant 1/n, \\ 0, & \text{if } U > 1/n. \end{cases}$$

Compute $\mathbb{E}[\lim_{n\to\infty} X_n]$ and $\lim_{n\to\infty} \mathbb{E}[X_n]$. Are they equal and why?

(2) Let U be uniform on $[0, 1]$. Compute

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{nU \log(U)}{1 + n^2 U^2}\right]$$

and justify your answer.