Stochastic Process Problem Set 2

1 Dish Washer at Silliman

The dining hall at Silliman has a dish washer. Sometimes the dish washer would malfunction due to unknown reasons, so that students have to use paper plates and plastic knives and forks. According to the chef, the lifetime of a dish washer of this model is ten years; in the first five years, it needs fixing every 2.5 years on average, while in the second five years, it needs a fix every two years on average.

1. Can you model this scenario as a homogeneous Poisson process? How about an inhomogeneous Poisson process?

2. Students may wonder the probability that a dish washer of the same model only has a single fix within its lifetime. What is that probability according to your model?

2 Convergence of Random Variables

1. Let $(\mathbb{N}, 2^{\mathbb{N}}, P)$ be a probability space, where $\mathbb{N}$ (the set of natural numbers) is the fundamental set and the σ-field is the power set (i.e., the complete σ-field). Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables defined in this probability space, and $\{X_n\}$ converges to X in probability. Does $\{X_n\}$ converge to X almost surely? Justify your answer by presenting a proof sketch or a counterexample. You may want to come up with a few example probability measures and random variable sequences on $\mathbb{N}$ (e.g., $P(\{i\}) = \frac{1}{2^i}$ and $X_n = 1_{\{n\}}$) and test their convergence.

2. Let $\{X_n\}$ be a sequence of random variables defined on a common probability space and X_n has the distribution $N(0, 1/n)$, i.e., the normal distribution with zero mean and variance $1/n$. Does $\{X_n\}$ converge almost surely, in L^2, in probability, or in distribution? Justify your answer.

3. Suppose that $X_n \to c$ in distribution, where c is a real number. Does it always hold that $X_n \to c$ in probability? Justify your answer.

4. Suppose that $X_n \to c$ in probability, where c is a real number. Does it always hold that $E(X_n) \to c$? Justify your answer.

5. Suppose that $X_n \to X$ in distribution. Let $g: \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that $g(X_n) \to g(X)$ in distribution. (Hint: Skorokhod representation theorem.)

6. Suppose that continuous random variables $X_n \to X$ in distribution. Therefore by definition, the cumulative distribution function (CDF) of X_n (denoted by F_n) converges to that of X (denoted by F) at every continuous point of F. Does it always hold that the probability density function of X_n converges to that of X? Justify your answer.

7. Let U be a uniform random variable on $[0, 1]$. Let

$$X_n = 1_{\{U \in [k/2^m, (k+1)/2^m]\}},$$

where $m = \lfloor \log_2(n) \rfloor = \max\{N \in \mathbb{N}: N \leqslant \log_2(n)\}$ and $k = n - 2^m$. Does $\{X_n\}$ converge almost surely, in L^2, in probability, or in distribution? Justify your answer.

8. Let U be a uniform random variable on $[0,1]$. Define

$$X_n = \sqrt{n^2+n} \cdot 1_{\left\{U \in \left(\frac{1}{n+1}, \frac{1}{n}\right)\right\}}.$$

Does $\{X_n\}$ converge almost surely, in L^2, in probability, or in distribution? Justify your answer.

9. Let X be normally distributed as $N(0,\, 1)$. For every n, let $X_n = (-1)^n X$. Does $\{X_n\}$ converges to X almost surely, in L^2, in probability, or in distribution? Justify your answer.

10. For every n, let U_n be uniformly distributed on the set $\{1,2,3,\ldots,n\}$. For every $1 \leqslant k \leqslant n$, let

$$X_{n,k} = n^2 1_{\{U_n = k\}}.$$

Consider the sequence of random variables $\{X_{1,1},\ X_{2,1},\ X_{2,2},\ X_{3,1},\ X_{3,2},\ X_{3,3},\ \ldots,\ X_{n,1},\ X_{n,2}, \ldots, X_{n,n}, \ldots\}$. Does it converge almost surely, in L^2, in probability, or in distribution? Justify your answer.

3 Angry Parent

Lin's parent enjoys texting him frequently. Based on historical statistics, messages arrive according to a Poisson process with rate 1. Right after sending the message, Lin's parent will feel angry until receiving Lin's reply. Lin's parent is looking at the mobile phone all the time. However, Lin is so busy that he can only have a look at his mobile phone every M time units. If he finds any new message coming from his parent, he will reply immediately and we assume that this does not cost any time. What is the limiting (long-term average) time fraction that his parent is angry?

4 Random Walk

Let $\{X_n\}$ be a sequence of i.i.d. random variables that can only take values from $\{-1,0,1\}$. Suppose that $E[X_n] = 0$. Let $S_n = \sum_{i=1}^n X_n$. This can be viewed as a random walk on integers; S_n is the position at time n and at each step, one can either move left ($X_n = -1$), stay ($X_n = 0$), or move right ($X_n = 1$). Let a and b be integers, where $a > 0$ and $b < 0$. Define

$$J = \min\{n \colon S_n \leqslant b \text{ or } S_n \geqslant a\}.$$

1. Is J a stopping time? Justify your answer.

2. Find all possible values of S_J.

3. Define $p = P(S_J \geqslant a)$. Find $E[S_J]$ in terms of a, b, and p.

4. Apply Wald's equation to $E[S_J]$ and find p.

You may want to rethink this problem after learning about the Markov chain.