Stochastic Process Problem Set 3

1 Handsome Dan with Hats

Handsome Dan has a big big house with two doors, the east and west doors. Dan has six hats and it places some of them at the east door and some at the west door. Every afternoon, it goes out and takes a walk along the Hillhouse Avenue and then comes back. It goes out from the west or east door equally likely and comes back to either doors also equally likely. When it goes out, if there is a hat at the door, it will put on the hat; otherwise, it will have a walk without a hat. When it comes back, it will put the hat (if any) at the door that it comes back to. We are interested in the number of hats at the west door every morning. Model it as a Markov chain and find the transition matrix and the steady state distribution.

2 Random Walk Revisited

In Problem 4 of Problem Set 2, we apply Wald's equation to $E[S_J]$ and find p. However, in order to apply Wald's equation, we have to show that $E[|J|]<\infty$. Show that $E[|J|]<\infty$ if $P(X_n=1)=q>0$. (Note that if $q=0$, it will be stuck at 0 forever and in this case, $E[|J|]=\infty$ and Wald's equation does not hold!) You may want to follow the steps listed below.

1. Show that $\{S_n : n\in\mathbb{N}\}$ is a Markov chain. Find the transition matrix.

2. Use the knowledge about Markov chains and show that for all $t=1,2,3,\ldots$

$$P(J>t(a-b))\leqslant(1-q^{a-b})^t.$$

3. Show that

$$E[J]=\sum_{k\geqslant 0}P(J>k)<\infty.$$

3 Classification of States

Let the Markov chain $X=\{X_n : n\geqslant 0\}$ have the state space $I=\{1,2,3,\ldots,7\}$. The transition matrix is

$$\mathbf{P}=\begin{pmatrix} 0.4 & 0 & 0.2 & 0.2 & 0.2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0.3 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0.6 \end{pmatrix}.$$

Determine if each state is transient, null recurrent or positive recurrent and find its period.

4 Fair Coin

Let's toss a fair coin repeatedly. Find the mean number of tosses until the first occurrence of three consecutive heads (HHH). What about the mean number of tosses until the first occurrence of THT? You may want to write a program to gain some intuition from simulation results before computing it analytically.