

CALIBRATION IN CIRCULAR ULTRASOUND TOMOGRAPHY DEVICES

Reza Parhizkar, Amin Karbasi and Martin Vetterli

School of Computer and Communication Sciences
Ecole Polytechnique Fédéral de Lausanne (EPFL), CH-1015, Switzerland

ABSTRACT

We consider the position calibration problem in circular tomography devices, where sensors deviate from a perfect circle. We introduce a new method of calibration based on the time-of-flight measurements between sensors when the enclosed medium is homogeneous. Bounds on the reconstruction errors are proven and results of simulations mimicking a scanning device are presented.

Index Terms— Calibration, Ultrasound Tomography, Matrix Completion, Multi-dimensional Scaling

1. INTRODUCTION

Ultrasound tomography aims at recovering certain features of a medium, based on measurements obtained by sending ultrasound signals through it. To this end, one requires to have a reliable setup for obtaining the measurements, a proper forward model imitating the measurement setup and an accurate inverse model based on which the characterizations of the medium can be estimated. Often, the forward model is used in the inverse process as well [1].

One of the key requirements of the methods for solving the forward and inverse problems is to have a good estimate of the positions of the sensors in the measurement setup. In order to associate the measurements to the values of the ultrasound field, the tomography model must be calibrated with the exact sensor locations prior to the experiment.

In this paper, we assume that the ultrasound tomography device has a circular shape, in which the sensors are placed approximately on the interior boundary of a ring surrounding the object to be scanned. This model is used for wind tomography [2] and breast cancer detection [3, 4].

In order to estimate the sensor positions, we use the time-of-flight (ToF) of ultrasound signals, defined by the time taken by an ultrasound wavefront to travel from a transmitter to a receiver. The position calibration is carried out using the ToF measurements between sensors in a homogeneous medium. However, there are a number of challenges in this work, namely,

- The ToF matrices obtained in a practical setup have missing entries.
- The measured entries of the ToF matrices are corrupted by noise.
- There is an unknown time delay added to the measurements.

If one had the complete and noiseless ToF matrix without time delay, the task of finding the exact positions would be very simple. This problem has been addressed in the literature as multi-dimensional

scaling (MDS) [5]. Nevertheless, the ToF matrix in a practical setup is never complete and many of the time-of-flight values are missing.

In general, it is a difficult task to infer missing entries of a matrix. However, it has recently been demonstrated that if the matrix has low rank, a small random subset of its entries allows to reconstruct it exactly. This result was first proved by Candès and Recht who analyzed a convex relaxation of this low-rank matrix completion problem [6]. More recently, an alternative approach using a combination of spectral techniques and manifold optimization was introduced in [7]. This algorithm used in our work is referred as OPTSPACE and has been shown to be stable under noisy measurements [8]. We show that a modified version of the ToF matrix has low rank and its missing entries can be accurately estimated using OPTSPACE. We state theoretical bounds on the performance of our proposed method under mild assumptions.

This paper is organized as follows. In Section 2, we define the problem statement and difficulties of calibration in circular devices. In Section 3, we introduce our main results on the error bounds for the calibration, and finally in Section 4, results of simulations for validating the proposed method are presented.

2. CIRCULAR TOMOGRAPHY

The focus of this paper is ultrasound tomography with circular apertures. In this setup, n ultrasound transducers are installed on the interior edge of a circular ring. The transducers are capable of both transmitting and receiving ultrasound signals. The general configuration for such a tomography device is depicted in Fig. 1. Each transducer is fired in turn while the rest record the scattered field reaching the ring. Using these measurements, one is interested in finding the sound speed distribution or sound attenuation inside the object surrounded by the ring.

2.1. Homogeneous Medium

If the medium inside the ring is homogeneous with constant sound speed c_0 , having the ToFs between sensors, one can construct a distance matrix \mathbf{D} consisting of the mutual distances between sensors as

$$\mathbf{D} = [d_{i,j}] = c_0 \mathbf{T}, \quad \mathbf{T} = [t_{i,j}], \quad i, j \in [n],$$

where $t_{i,j}$ is the ToF between sensors i and j . Note that the only difference between \mathbf{T} and \mathbf{D} , is the constant c_0 . Knowing temperature and characteristics of the homogeneous medium inside the ring, one can accurately estimate c_0 . Thus, we will assume that c_0 is a-priori known for the calibration and in the sequel will focus on the distance matrix \mathbf{D} rather than the actual measured matrix \mathbf{T} .

Since the enclosed medium is homogeneous, matrix \mathbf{D} is symmetric with zeros on the diagonal. Even though the distance matrix is full rank in general, a simple point-wise transform of its entries

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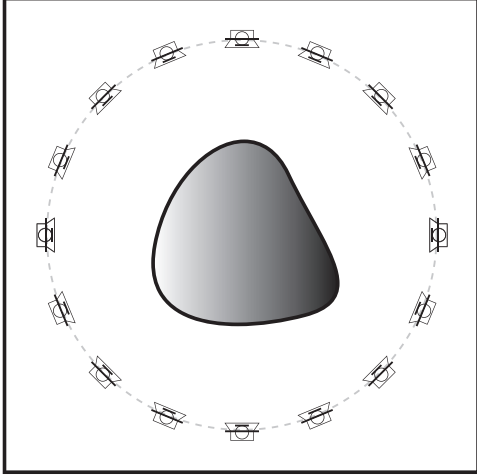


Fig. 1. Circular setup for ultrasound tomography considered in this work. Ultrasound transducers are distributed on the edge of a circular ring and the object with unknown characteristics is put inside. Sources and receivers are colocated.

leads to a low rank matrix. More precisely we can prove the following lemma

Lemma 1. If one constructs the squared distance matrix \bar{D} as

$$\bar{D} = [d_{i,j}^2], \quad i, j \in [n],$$

then the matrix \bar{D} has rank at most 4 and if the sensors are on a circle, the rank is exactly 3.

Proof. The proof for the general case is provided in [5] and the proof for circular case can be found in our technical report [9]. \square

2.2. Time of Flight Estimation

In order to estimate the exact time of flights, one needs to compare the received signal with the transmitted signal and use existing methods (e.g. cross correlation based methods) to find the relative delay between them. However, in our case, the lack of a reference signal leaves no choice than computing the *absolute ToFs*. One way to find the absolute ToFs is to probe the received signal and define the ToF as the time instance at which the received signal power exceeds some predefined threshold. This is also known as the first arrival method. However, using this method, one cannot retrieve the information about the time origin in the signal. Together with the transitional behaviour of the sensors, this causes an unknown delay added to the actual ToFs.

Moreover, because of the limited beam width of ultrasound transducers deployed on the circle, and also the late response of the transducers, it is not possible to obtain reliable measurements for the sensors close to each other. Thus, numbering the sensors from 1 to n , in the ToF matrix T (equivalently D), we do not have measurements on specific bands of the matrix. According to Fig. 2, we will assume that each sensor does not have ToF measurement of the sensors which lie in a radius of δ_n from it. We will call these entries as *structured missing entries*.

Further, during the measurement and ToF estimation procedure, some pairs of sensors give outliers. This can be caused by measurement noise, or the ToF estimation algorithm for finding the first arrivals. One can define a smoothness criteria in the ToF matrix and

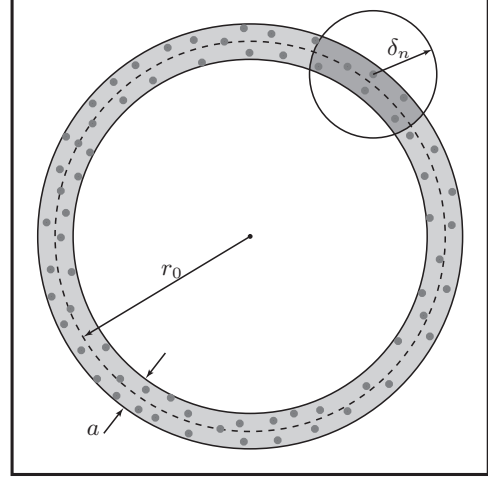
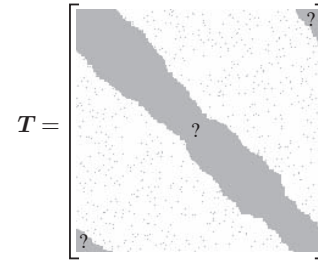


Fig. 2. Sensors are uniformly distributed on a circular ring of width a . The beam width and transient response of the transmitter causes the neighbouring sensors not to have reliable ToF measurements. Sensors lying in the dark gray part will not have a distance measurement from the centred sensor.

remove the entries which do not satisfy this criteria. We will refer to these entries as *random missing entries*. Eventually, an instance of the ToF matrix will look like



where, the gray parts represent missing entries. The above mentioned problems result in an incomplete matrix T , which cannot be used for position reconstruction, unless the unknown time delay is removed from the measurements and the unknown entries are estimated.

3. POSITION CALIBRATION

In this paper, we assume that the time delay is known and subtracted from the ToF measurements. We have provided a method in [9] for heuristically finding the time delay in a case where the sensors are distributed exactly on a circle. The method is also applicable to the current model. Here, in our model we assume that sensor positions deviate from a perfect circle. More precisely, we will assume that the sensors are uniformly distributed on the area between two circles of radii $r_0 - a/2$ and $r_0 + a/2$ (the light gray ring in Fig. 2).

In order to incorporate the structured missing entries, we assume that measurements between sensors of distance less than δ_n are missing. The number of structured missing entries depends on δ_n^2 . We are interested in a regime where we have a small number of structured missing entries in the large systems limit. Accordingly, we will assume that δ_n of interest is $\delta_n = \Theta(r\sqrt{\log n/n})$. A random set of structured missing indices $S \subseteq [n] \times [n]$ is defined from $\{x_i\}$ and

δ_n , by

$$S = \{(i, j) : d_{i,j} \leq \delta_n \text{ and } i \neq j\},$$

where $d_{i,j} = \|\mathbf{x}_i - \mathbf{x}_j\|$. Then, the structured missing entries are denoted by a matrix

$$\mathbf{D}_{i,j}^s = \begin{cases} \mathbf{D}_{i,j} & \text{if } (i, j) \in S \\ 0 & \text{otherwise.} \end{cases}$$

Note that the matrix $\mathbf{D}^s = \mathbf{D} - \mathbf{D}^s$ captures the noiseless distance measurements that are not affected by structured missing entries. Next, to model the noise we add a random noise matrix \mathbf{Z}^s

$$\mathbf{Z}_{i,j}^s = \begin{cases} \mathbf{Z}_{i,j} & \text{if } (i, j) \in S^\perp \\ 0 & \text{otherwise,} \end{cases}$$

where S^\perp denotes the complementary set of S .

Finally to model the random missing entries, we assume that each entry of $\mathbf{D} - \mathbf{D}^s + \mathbf{Z}^s$ is sampled with probability p_n . In the calibration data, we typically have a small number of random missing entries. Hence, we assume that $p_n = \theta(1)$. Let $E \subseteq [n] \times [n]$ denote the subset of indices which are not erased by random missing entries. Then a projection $\mathcal{P}_E : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n}$ is defined as

$$\mathcal{P}_E(\mathbf{M})_{i,j} = \begin{cases} \mathbf{M}_{i,j} & \text{if } (i, j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

We denote the observed measurement matrix by

$$\mathbf{N}^E = \mathcal{P}_E(\mathbf{D} - \mathbf{D}^s + \mathbf{Z}^s).$$

Goal: Given the observed matrix \mathbf{N}^E and the missing indices $S \cup E^\perp$, we want to estimate the positions matrix $\mathbf{X} \in \mathbb{R}^{n \times 2}$.

In order to achieve this goal, first, we need to infer the missing entries of \mathbf{N}^E , and then estimate the sensor positions given approximate pairwise distances. For the former, we use OPTSPACE algorithm [8] taking $(-\mathbf{D}^s + \mathbf{Z}^s)$ as the new noise matrix and rank 4 property for $\bar{\mathbf{D}}$. For the latter, we use the MDS algorithm [5]. The following two theorems, provide error bounds on the reconstruction in both steps.

Theorem 1. Assume n sensors are distributed uniformly at random on a circular ring of width a with central radius r_0 as in Fig. 2. The resulting distance matrix \mathbf{D} is corrupted by structured missing entries \mathbf{D}^s and measurement noise \mathbf{Z}^s . Further, the entries are missing randomly with probability p_n . Let $\mathbf{N}^E = \mathcal{P}_E(\mathbf{D} - \mathbf{D}^s + \mathbf{Z}^s)$ denote the observed matrix. Define $\bar{\mathbf{D}}$ as the squared distance matrix. Assume $\delta_n = \delta r_0 \sqrt{\log n/n}$ and $p_n = p$. Then, there exist constants C_1 and C_2 , such that the output of OPTSPACE $\hat{\bar{\mathbf{D}}}$ achieves

$$\frac{1}{n} \|\bar{\mathbf{D}} - \hat{\bar{\mathbf{D}}}\|_F \leq C_1 \left(\sqrt{\frac{\log n}{n}} \right)^3 + C_2 \frac{\|\mathcal{P}_E(\mathbf{Y}^s)\|_2}{pn}, \quad (1)$$

with probability larger than $1 - n^{-3}$, where $\mathbf{Y}_{i,j}^s = \mathbf{Z}_{i,j}^2 + 2\mathbf{Z}_{i,j}\mathbf{D}_{i,j}$ and

$$C_1 = c\delta^3(r_0 + a)^2.$$

We do not assume a prior distribution on \mathbf{Z} , and the above theorem is stated for any general noise matrix \mathbf{Z} .

For evaluating the error on the reconstruction of the positions, we need to define a notion of distance for the error. Since, the MDS algorithm can find the positions up to a rigid transformation (rotation, translation and reflection), it is not possible to compare the

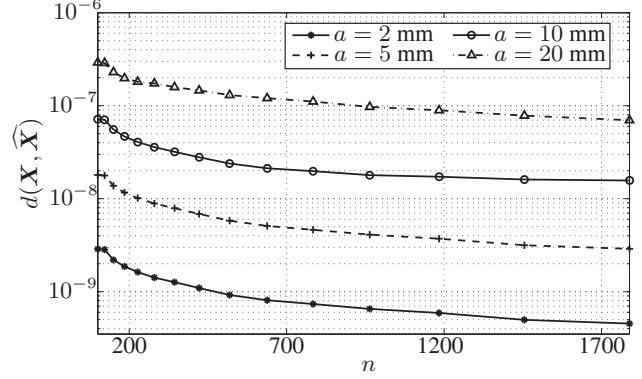


Fig. 3. Error in position estimation in noiseless case for different values of a .

results of MDS directly with the exact positions. Thus, as defined in [10], we will use the matrix

$$\mathbf{L} = \mathbb{I}_n - (1/n)\mathbb{1}_n\mathbb{1}_n^T, \quad (2)$$

where $\mathbb{1}_n \in \mathbb{R}^n$ is the all ones vector, and \mathbb{I}_n is the identity matrix. Using the above definition, one can define the error of the position reconstruction as

$$d(\mathbf{X}, \widehat{\mathbf{X}}) = \frac{1}{n} \|\mathbf{L}\mathbf{X}\mathbf{X}^T\mathbf{L} - \mathbf{L}\widehat{\mathbf{X}}\widehat{\mathbf{X}}^T\mathbf{L}\|_F. \quad (3)$$

Theorem 2. Applying multidimensional scaling algorithm on $\hat{\bar{\mathbf{D}}}$, the error on the resulting coordinates will be bounded as

$$d(\mathbf{X}, \widehat{\mathbf{X}}) \leq C_1 \left(\sqrt{\frac{\log n}{n}} \right)^3 + C_2 \frac{\|\mathcal{P}_E(\mathbf{Y}^s)\|_2}{pn}, \quad (4)$$

with probability larger than $1 - n^{-3}$.

The proofs for the theorems are omitted for brevity and interested readers are referred to our technical report [9].

The first term in (4) is in fact due to the structured missing entries and the second term is caused by the noise in the measurements. Thus, in a noiseless scenario, the position reconstruction error will decrease fast as n grows, whereas in a noisy case, the error rate also depends on the distribution of the noise matrix.

4. EXPERIMENTAL RESULTS

In order to evaluate the performance of the calibration method, three sets of experiments are done. First, the distance matrix is assumed noiseless, and the position estimation error is derived for different values of n and the ring width a . The value of r_0 is set to 10 cm, on average 5 percent of entries are missing randomly, and δ in Theorem 1 is assumed to be 1. The results are reported in Fig. 3. As expected from Theorem 2, the general trend in all curves is that the error decreases as n grows. Moreover, the larger a is, the bigger is the reconstruction error, which is also compatible with the results of Theorem 2.

To examine the stability of the estimation algorithm under noise, we set the values of a to 1 cm, δ to 1, r_0 to 10 cm, and the percentage of random missing entries to 5. We added to each entry of the distance matrix \mathbf{D} a centred white Gaussian noise of different standard deviations. The results are depicted in Fig. 4. As the variance of the

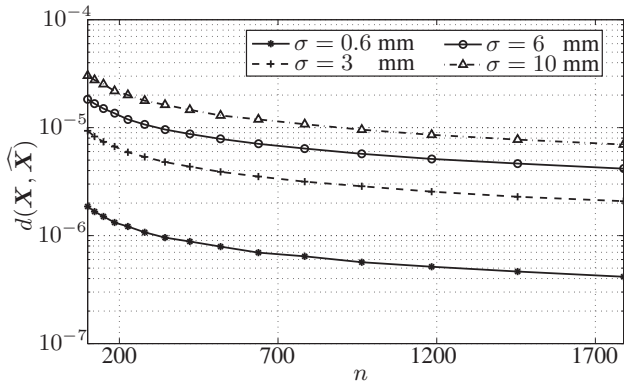


Fig. 4. Error in position estimation for the case with centred white Gaussian noise of different standard deviations, σ .

noise increases, the position estimation error grows, but in general the error decreases for larger n .

Moreover, to show the importance of calibration in an ultrasound scanning device, a simple experiment is performed. If the ToF measurements correspond to the exact positions of sensors, reconstruction of water will lead to a homogeneous region with values equal to the water sound speed, whereas wrong assumption on the sensor positions causes the inverse method to give unrelated values as the sound speed to compensate the effect of position mismatch.

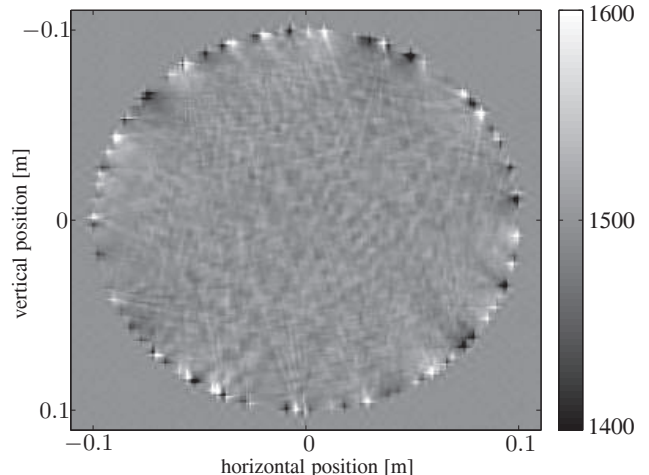
In a simple experiment, we simulated the reconstruction of water sound speed using the ToF measurements. In the simulation, 200 sensors are distributed around a circle with radius $r_0 = 10$ cm, and they deviate on average 0.5 mm from the circumference. In Fig. 5(a), it is assumed that the sensors are exactly on a circle, while in Fig. 5(b), we put the output of the calibration as the sensor positions. Clearly an incorrect assumption on the sensor positions has a large effect in the reconstruction of the medium.

5. CONCLUSION

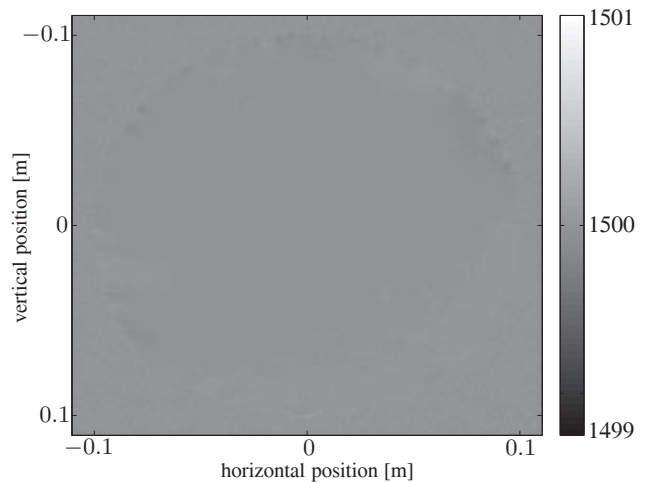
The simulation results verify the error bounds found for the estimation of ultrasound sensor positions. We observe that the method is robust under noise effect as well. One can compare the considered problem with the classical sensor localization problem, where local connectivity information between sensors are missing.

6. REFERENCES

- [1] F. Natterer, "Acoustic mammography in the time domain," Tech. Rep., University of Münster, 2008.
- [2] I. Jovanovic, L. Sbaiz, and M. Vetterli, "Acoustic tomography for scalar and vector fields: theory and application to temperature and wind estimation," *Jour. of Atmos. and Oceanic Tech.*, vol. 26, no. 8, pp. 1475–1492, 2009.
- [3] N. Duric, P. Littrup, L. Poulou, A. Babkin, A. Pevzner, E. Holsapple, Rama O., and C. Glide, "Detection of breast cancer with ultrasound tomography: First results with the computed ultrasound risk evaluation (cure) prototype," *Med. Phys.*, vol. 34, no. 2, pp. 773–785, 2007.
- [4] F. Simonetti, L. Huang, and N. Duric, "On the spatial sampling of wave fields with circular ring apertures," *Jour. of App. Phys.*, vol. 101, no. 8, pp. 083103–083103–6, 2007.
- [5] P. Drineas, A. Javed, M. Magdon-Ismail, G. Pandurangant, R. Virrankoski, and A. Savvides, "Distance matrix reconstruc-



(a)



(b)

Fig. 5. Simulation results for a simple reconstruction of water when (a) the sensors are assumed to be on a perfect circle, and (b) when the inversion method is first calibrated with the estimated sensor positions.

- tion from incomplete distance information for sensor network localization," in *Sensor and Ad Hoc Comm. and Networks*, 2006, vol. 2, pp. 536–544.
- [6] E. J. Candès and B. Recht, "Exact matrix completion via convex optimization," *Found. of Comput. Math.*, vol. 9, pp. 717–772, 2009.
- [7] R. H. Keshavan, M. Montanari, and S. Oh, "Matrix completion from a few entries," *IEEE Trans. Inform. Theory*, 2010, arXiv:0901.3150.
- [8] R. H. Keshavan, A. Montanari, and S. Oh, "Matrix completion from noisy entries," in *Advances in Neural Information Processing Systems*, 2009.
- [9] R. Parhizkar, A. Karbasi, S. Oh, and M. Vetterli, "Calibration in ultrasound tomography devices using structured matrix completion," <https://infoscience.epfl.ch/record/153381>.
- [10] A. Karbasi, S. Oh, R. Parhizkar, and Vetterli M., "Ultrasound tomography calibration using structured matrix completion," in *20th Intern. Cong. on Acoustics*, 2010.