Stochastic Process Problem Set 4

1 Random Walk on Integers

Let $\{X_n : n \in \mathbb{N}\}$ be i.i.d. random variables that only take values from $\{-1, 1\}$. Let $p = P(X_n = -1)$ and $S_n = \sum_{i=1}^{n} X_i$. Thus $\{S_n : n \in \mathbb{N}\}$ is a Markov chain whose state space is $\mathbb{Z}$, the set of integers.

1. Find the probability

$$p_{00}^{(n)} = P(S_{n+j} = 0 | S_j = 0).$$

2. Determine whether each state is transient, null recurrent or positive recurrent by computing $\sum_{n=1}^{\infty} p_{00}^{(n)}$ and using the famous Stirling's approximation:

$$\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1.$$

(Hint: discuss the two cases $p = 1/2$ and $p \neq 1/2$.)

2 Random Knight

Consider a knight sitting on the lower left corner square (denoted by "a1") of an 8×8 chess board with the square "e4" removed. The knight has residual frog-like tendencies, left over from an old spell an older witch cast upon him. So he performs a random walk on the chess board, at each time choosing a random move uniformly distributed over the set of his possible knight moves (any move to "e4" is impossible). What is the expected time until he first returns to the lower left corner square?

3 Teleportation

An ant walks randomly on the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. The position of the ant at time t is denoted by $X_t \in \mathbb{N}$, where $t = 1, 2, 3, \ldots$. This ant moves up with probability $1/2$, down with probability $3/8$, and is teleported to 0 with probability $1/8$; *i.e.*,

$$X_{t+1} = \begin{cases} X_t + 1 & \text{with probability } 1/2, \\ \max\{X_t - 1, 0\} & \text{with probability } 3/8, \\ 0 & \text{with probability } 1/8. \end{cases}$$

1. Can this be modeled as a Markov chain? If so, find the steady state distribution.

2. Is there any absorbing state in this Markov chain?

3. Find the mean recurrence time τ_n for some state $n \in \mathbb{N}$.

4. Find a function of n, denoted by $f(n)$, such that $\tau_{0,n} = \Theta(f(n))$, where $\tau_{0,n}$ is the expected time to get to state n from state 0. The notation $\tau_{0,n} = \Theta(f(n))$ means that there exists two positive constants C_1 and C_2 such that

$$C_1 f(n) \leqslant \tau_{0,n} \leqslant C_2 f(n), \forall n \in \mathbb{N}.$$

(Hint: find the relationship between τ_n and $\tau_{0,n}$)

4 Handsome Dan's Cookies

Guess what? Handsome Dan really loves eating cookies! Every time when it has eaten n cookies, it will eat another $(n+1)$ cookies on average! That is to say, if we let N denote the total number of cookies that Handsome Dan eats, then we know that

$$E[N|N \geqslant n] = 2n+1.$$

For each $n \geqslant 1$, find the probability that Handsome Dan stops eating more cookies when it has already eaten n cookies; *i.e.*, find the conditional probability

$$P(N=n|N \geqslant n).$$

You may want to follow the steps listed below:

1. Show that $P(N \geqslant n)$ is nonzero for all $n \geqslant 0$.
2. Show that for all n,
$$\sum_{k=n}^{\infty} kP(N=k) = (2n+1)P(N \geqslant n).$$
3. Find $P(N=n|N \geqslant n)$.

5 Martingale

We know that there can be at most $\binom{n}{2} = \frac{n(n-1)}{2}$ edges among n vertices. Let X_i be the indicator random variable for the event that the ith edge exists; *i.e.*, X_i is one if the ith edge exists and it is zero otherwise. So we have a sequence of random variables

$$\left\{X_i: 1 \leqslant i \leqslant \binom{n}{2}\right\}.$$

They may not be i.i.d. Let χ be the chromatic number of the undirected graph comprised of the n vertices, which is the minimum number of colors necessary when we assign a color to each vertex while neighbors have different colors. For example, consider the chromatic number of the undirected ring graphs $R_k = (V_k, E_k)$, where the vertex set $V_k = \{1, 2, 3, ..., k\}$ and the edge set $E_k = \{(j, (j+1) \bmod k): j = 1, 2, ..., k\}$; then we know that the chromatic number of R_2 is 2 and the chromatic number of R_3 is 3 (having two colors is insufficient for R_3), but the chromatic number of R_4 is only 2.

Let's come back to the chromatic number of the undirected graph comprised of the n vertices. Let

$$C_i = E[\chi|X_1, X_2, ..., X_i].$$

Show that

$$\left\{C_i: 1 \leqslant i \leqslant \binom{n}{2}\right\}$$

is a martingale; *i.e.*, show that for all i, $E[|C_i|] < \infty$ and

$$E[C_{i+1}|X_1, ..., X_i] = C_i.$$