Outline of Third Recitation Session

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1 Examples about Convergence of Random Variables

1. Convergence in L^p but not almost sure:

$$X_n = 1_{[0,1]}(2^m U - k),$$

where U is uniformly distributed on $[0,1]$ and $n = 2^m + k$ with $0 \leqslant k < 2^m$.

2. Almost sure convergence but not in L^p:

$$X_n = n 1_{[0,1/n]}(U),$$

where U is uniform distributed on $[0,1]$.

3. Convergence in probability but neither almost sure nor in L^p:

$$X_{n,k} = 4^n 1_{[k/2^n,(k+1)/2^n]}(U),$$

where $k = 0, 1, ..., 2^n - 1$.

4. Convergence in distribution but not in probability: define $\Omega = \{1,2,3,4\}$,

$$X_n(1) = X_n(2) = 1, X_n(3) = X_n(4) = 0,$$

$$X(1) = X(2) = 0, X(3) = X(4) = 1.$$

5. Convergence in L^1 but not in L^2:

$$X_n = n^{2/3} 1_{U \leqslant 1/n}.$$

6. Convergence in L^q $(0 < q < p)$ but not in L^r $(r \geqslant p)$:

$$X_n = n^{1/p} 1_{U \leqslant 1/n}.$$

One can use this to construct a counterexample for convergence in L^q but not in L^r for any $0 < q < r$.

7. Almost sure convergence but Borel-Cantelli Lemma is not satisified (same example as in the second item):

$$X_n = n 1_{[0,1/n]}(U).$$

2 Quick Review for σ-Fields and Random Variables

1. The motivation of defining a random variable as a $\mathcal{A}/\mathcal{B}(\mathbb{R})$-measurable function. Why not use other σ-fields on $\mathbb{R}$? I will explain this from the perspective of expectation, which plays a central role in the measure-theoretic probability theory.

2. I will explain the following problem:

Problem 1. Let $\Omega = \{a, b, c\}$ and $A = \{\{b, c\}, \{a, b\}\}$. We define three random variables X, Y, Z as follows:

ω	X	Y	Z
a	1	1	1
b	1	2	7
c	2	2	4

a. Which of the random variables are $\sigma(A)$-measurable?

b. Find $\sigma(Z)$ and $\sigma(Y)$. Is Y, $\sigma(Z)$-measurable? Is X, $\sigma(Y)$-measurable?