Outline of Third Recitation Session

BY LIN CHEN

1 Examples about Convergence of Random Variables

1. Convergence in $L^p$ but not almost sure:

   $$X_n = 1_{[0,1]}(2^m U - k),$$

   where $U$ is uniformly distributed on $[0, 1]$ and $n = 2^m + k$ with $0 \leq k < 2^m$.

2. Almost sure convergence but not in $L^p$:

   $$X_n = n 1_{[0,1/n]}(U),$$

   where $U$ is uniformly distributed on $[0, 1]$.

3. Convergence in probability but neither almost sure nor in $L^p$:

   $$X_{n,k} = 4^n 1_{[k/2^n,(k+1)/2^n]}(U),$$

   where $k = 0, 1, ..., 2^n - 1$.

4. Convergence in distribution but not in probability: define $\Omega = \{1, 2, 3, 4\}$,

   $$X_n(1) = X_n(2) = 1, X_n(3) = X_n(4) = 0,$$

   $$X(1) = X(2) = 0, X(3) = X(4) = 1.$$

5. Convergence in $L^1$ but not in $L^2$:

   $$X_n = n^{2/3} 1_{U \leq 1/n}.$$

6. Convergence in $L^q$ ($0 < q < p$) but not in $L^r$ ($r \geq p$):

   $$X_n = n^{1/p} 1_{U \leq 1/n}.$$

   One can use this to construct a counterexample for convergence in $L^q$ but not in $L^r$ for any $0 < q < r$.

7. Almost sure convergence but Borel-Cantelli Lemma is not satisfied (same example as in the second item):

   $$X_n = n 1_{[0,1/n]}(U).$$

2 Quick Review for $\sigma$-Fields and Random Variables

1. The motivation of defining a random variable as a $A/B(\mathbb{R})$-measurable function. Why not use other $\sigma$-fields on $\mathbb{R}$? I will explain this from the perspective of expectation, which plays a central role in the measure-theoretic probability theory.
2. I will explain the following problem:

**Problem 1.** Let $\Omega = \{a, b, c\}$ and $A = \{\{b, c\}, \{a, b\}\}$. We define three random variables $X, Y, Z$ as follows:

<table>
<thead>
<tr>
<th>$\omega$</th>
<th>$X$</th>
<th>$Y$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$b$</td>
<td>1</td>
<td>2</td>
<td>7</td>
</tr>
<tr>
<td>$c$</td>
<td>2</td>
<td>2</td>
<td>4</td>
</tr>
</tbody>
</table>

a. Which of the random variables are $\sigma(A)$-measurable?

b. Find $\sigma(Z)$ and $\sigma(Y)$. Is $Y$, $\sigma(Z)$-measurable? Is $X$, $\sigma(Y)$-measurable?