1 The Tipping Point

In most given systems, there is an inherent structure where objects (nodes) interact with each other (through links or edges). Critical to understanding how little things can make a big difference in the formation of random networks is the concept of tipping points. In his national best-selling book "The Tipping Point: How Little Things Can Make a Big Difference", Malcolm Gladwell defines a tipping point as that magic moment when an idea, trend, or social behavior crosses a threshold, tips, and spreads like wildfire. Just as a single sick person can start an epidemic of the flu, so too can a small but precisely targeted push cause a fashion trend, the popularity of a new product, or a drop in the crime rate.

Various disciplines provide us with alternate ways of approaching the study of tipping points. For instance, physicists study tipping points through phase transitions. For example in the case of water, if its temperature stays above zero degrees and less than one hundred degrees Celsius it remains a liquid. But once it crosses those thresholds it changes its state: below 0°C it becomes ice, and above 100°C it suddenly changes to steam. As another example, in sociology, a tipping point is a point in time when a group rapidly and dramatically changes its behavior by widely adopting a previously rare practice.

In this lecture, we study an elegant tipping point taking place in what’s called the branching process, first proposed by Galton and Watson in 1873. The model considers each individual as a node in the tree, and the root of the tree corresponds to the ancestor of the whole population. Each node is connected to its parent as well as child, and each individual gives birth to a certain number of children. The Galton-Watson branching process discussed in this lecture is a model for heredity and attempts to shed light on the survival probability of the population.

1.1 Probability Generating Function

We will be using the concept of Generating Functions (GF), so it would help to recall some basic definitions and properties.

If a discrete random variable $X$ takes values from $\{0,1,2,...\}$ and has a probability distribution $p_k = \Pr(X = k)$ then its generating function is

$$
\phi_X(s) = \mathbb{E}[s^X] = \sum_{k=0}^{\infty} s^k p_k = p_0 + p_1 s + p_2 s^2 + ... 
$$

(1)

for any value $0 \leq s \leq 1$. Note that

$$
\phi_X(1) = \sum_k p_k = 1
$$

(2)

which means that $s = 1$ is a fixed point. It also follows that:

$$
\phi'(1) = \mathbb{E}[X]
$$

(3)
1.2 Branching Model

The branching process is a model of heredity. It starts with one node. Then, each individual has a probability \( p_k \) of having \( k \) offsprings independent from anyone else. Let \( \xi \) be a random variable denoting the number of offsprings of an individual, i.e., \( p_k = \Pr(\xi = k) \). Furthermore, let \( T_n \) be the total number of individuals born up to and including generation \( n \), and \( X_i \) be the total number of individuals in generation \( i \) (assuming that the process starts with a single individual, \( X_0 = 1 \)). The question we would like to answer is whether or not the population \( T \) (which is a tree) survives. In other words, we would like to find the probability that the tree \( T \) would be a finite tree

\[
\Pr(\text{extinction}) = p_{ex} = \Pr(|T| < \infty) \tag{4}
\]

The population tree can be divided into subtrees, and the subtrees all follow a similar distribution as the main tree (c.f., Fig. 1).

![Figure 1](image)

**Figure 1:** A branching process with 2 generations.

Given that \( X_1 = k \) we have,

\[
\Pr(\text{extinction}|X_1 = k) = \Pr(|T_1| < \infty, |T_2| < \infty, \ldots, |T_k| < \infty) = p(|T| < \infty)^k = p_{ex}^k.
\]

So, the probability of extinction is

\[
p_{ex} = \sum_{k=0}^{\infty} \Pr(|T| < \infty|X_1 = k).p_k = \sum_{k=0}^{\infty} p(|T| < \infty)^k.p_k = \sum_{k=0}^{\infty} p_{ex}^k.p_k.
\]

As a result, we have

\[
p_{ex} = \phi_\xi(p_{ex}).
\]
So, \( p_{\text{ex}} \) is also a fixed point of \( \phi_\xi(s) \).

Let \( p_{\text{ex}}^{(n)} \) denote the probability that the population go extinct at generation \( n \) or before, i.e.,
\[
p_{\text{ex}}^{(n)} = p(X_n = 0)
\]
Note that if \( X_n = 0 \Rightarrow X_{(n+1)} = 0 \), as \( \{X_n = 0\} \subseteq \{X_{(n+1)} = 0\} \). So, clearly we have
\[
p_{\text{ex}}^{(0)} \leq p_{\text{ex}}^{(1)} \leq p_{\text{ex}}^{(2)} \leq \ldots \leq p_{\text{ex}}^{(n)},
\]
which implies that
\[
\lim_{n \to \infty} p_{\text{ex}}^{(n)} = p_{\text{ex}}.
\]
Using the definition of the probability generating function defined we obtain
\[
\phi_n(s) = \mathbb{E}[s^{X_n}] = \mathbb{E}[\mathbb{E}[s^{X_n} | X_1]] = \sum_k \mathbb{E}[s^{X_n} | X_1 = k]p_k.
\]
Let \( \tilde{X}_n^i \) represent all children in the \( n \)th generation whose ancestors are from sub-tree \( T_i \). We can simplify the above expression as:
\[
\sum_k \mathbb{E}[s^{X_n} | X_1 = k]p_k = \sum_k \mathbb{E}[s^{\tilde{X}^1_n + \tilde{X}^2_n + \ldots + \tilde{X}^k_n}]p_k
\]
Note that \( \tilde{X}_n^i \) and \( X_{n-1} \) have the same distribution. So,
\[
\sum_k \mathbb{E}[s^{\tilde{X}^1_n + \tilde{X}^2_n + \ldots + \tilde{X}^k_n}]p_k = \sum_k \mathbb{E}[s^{X_{n-1}}]^k p_k = \sum_k \phi^{k}_{n-1}(s)p_k.
\]
Hence,
\[
\phi_n(s) = \sum_k \phi^{k}_{n-1}(s)p_k = \phi_\xi(\phi_{n-1}(s)).
\]
As a result
\[
p_{\text{ex}}^{(n)} = \phi_\xi(p_{\text{ex}}^{(n-1)}).
\]
Now, we prove that \( p_{\text{ex}} \) is the first fixed point. Assume that \( 0 \leq x^* \leq 1 \) is another fixed point, i.e., \( \phi_\xi(x^*) = x^* \). We show by induction that \( p_{\text{ex}}^{(n)} \leq x^* \) for any \( n \). The base case is trivial \( p_{\text{ex}}^{(0)} = 0 \leq x^* \). Assume that \( p_{\text{ex}}^{(n)} \leq x^* \). Since \( \phi_\xi(s) \) is an increasing function of \( s \) (its derivative is always positive), we have
\[
p_{\text{ex}}^{(n+1)} = \phi_\xi(p_{\text{ex}}^{(n)}) \leq \phi_\xi(x^*) = x^*
\]
This implies that \( p_{\text{ex}} = \lim_n p_{\text{ex}}^{(n)} \leq x^* \) is the first fixed point.

The generating function \( \phi_\xi(s) \) is not only non-decreasing, it is also convex for \( s \geq 0 \) (its second derivative is always non-negative):
\[
\phi''_\xi(s) = \sum_{k=2}^{\infty} p_k k(k-1)s^{k-2} \geq 0
\]
Note that if \( p_0 = 0 \) then \( p(\text{survive}) = 1 \). Otherwise, \( p_1 < 1 \) and we have 3 cases depending on the value of \( \mathbb{E}(\xi) \).
Case 1: $\mathbb{E}(\xi) < 1$

The function $\phi_\xi(s)$ is non-decreasing and convex (so, its slope cannot decrease). Note that the slope of $\phi_\xi(s)$ at $s = 1$ is $\phi'_\xi(1) = \mathbb{E}(\xi) < 1$. As a result, the function $\phi_\xi(s)$ and $y = s$ first intersect at $s = 1$.

![Figure 2: Plot of $\phi_\xi(s)$ vs $s$ for case 1.](image)

Case 2: $\mathbb{E}(\xi) = 1$

If $p_0 > 0$, then $p_1 < 1$. By $\mathbb{E}(\xi) = 1$ we know that $\exists k \geq 2$ such that $p_k > 0$. As a result $\phi_\xi(s)$ is strictly convex. Again, we see that $\phi_\xi(1) = 1$ is the first fixed point.

![Figure 3: Plot of $\phi_\xi(s)$ vs $s$ for case 2.](image)
**Case 3:** $\mathbb{E}(\xi) > 1$

If $p_0 > 0$ then $p_1 < 1$. Since $\mathbb{E}(\xi) > 1$, convexity implies that $\phi_\xi(s)$ has 2 fixed points in $[0, 1]$ and the smaller one is $p_{ex}$.

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**Figure 4:** Plot of $\phi_\xi(s)$ vs $s$ for case 3.